Counterexamples of functors that do not preserve parametric simple Ω-products

Kazuyuki Asada

1 Introduction

In [1], it is shown that all the polynomial functors—including (linearly-)initial algebra and final coalgebra construction—in a suitable model of parametric polymorphic lambda calculus preserve simple Ω-products. (Simple Ω-products are semantical counterparts of universal types $\forall\alpha.\sigma$.) In this manuscript, we give examples of functors that do not preserve simple Ω-products even in parametric setting. Especially, the continuation monad in Section 2 forms an example of models that are not models of $\lambda_c 2_\eta$-calculus [1]. Below, see ibid. for related notions.

2 Continuation Monad

The next example is given in a syntactic way (or, in term models).

Proposition 1 *Let R be the type $1+1$ in System F.*
Then the canonical term

$$\begin{array}{ll} \textit{from} & ((\forall\alpha.\alpha \to \alpha + R) \to R) \to R \\ \textit{to} & \forall\alpha.\,((\alpha \to \alpha + R) \to R) \to R \end{array}$$

is not an isomorphism (regardless of assuming parametricity). $\square$

Proof. Let c be the canonical term from

$$((\forall\alpha.\alpha \to \alpha + R) \to R) \to R$$

to

$$\forall\alpha.\,((\alpha \to \alpha + R) \to R) \to R.$$

Suppose that there is a term c^{-1} such that c is isomorphic up to parametricity with c^{-1} as the inverse. (Note that if c is isomorphic up-to-$\beta\eta$ then trivially is isomorphic up to parametricity.) Then our aim is to show absurdity.

$$\begin{tikzcd}
((\forall\alpha.\alpha \to \alpha + R) \to R) \to R \arrow[r, shift left=2, "c"] \arrow[r, phantom, "\cong?"] \arrow[d, shift left=2, "d^{-1}"] \arrow[d, phantom, "\cong"] & \forall\alpha.\,((\alpha \to \alpha + R) \to R) \to R \arrow[l, shift left=2, "c^{-1}"] \arrow[d, "(-)R"] \\
((1+R) \to R) \to R \arrow[u, shift left=2, "d"] \arrow[r, dotted, "f"] & ((R \to R + R) \to R) \to R
\end{tikzcd}$$

By usual parametric reasoning, $\forall\alpha.\alpha \to \alpha + R$ is isomorphic to $1 + R$, so there is the canonical term d and d^{-1} as above, and they are isomorphic up to parametricity. Then let f be the composition $((-)\,R) \circ c \circ d$.

Let $1'$ be the term

$$\Lambda\alpha.\lambda k : ((\alpha \to \alpha + R) \to R)\,.k\,(\lambda x_0 : \alpha.\mathrm{in}_1\,(k\,(\lambda x_1 : \alpha.\mathrm{in}_0\,(x_0))))$$

in $\forall\alpha.\,((\alpha \to \alpha + R) \to R) \to R$. (On the naming of "$1'$", note that we can define the Church numeral of 1 in $((1 + R) \to R) \to R$, and its image by $c \circ d$ is equal to the term

$$\Lambda\alpha.\lambda k : ((\alpha \to \alpha + R) \to R)\,.k\,(\lambda x_0 : \alpha.\mathrm{in}_1\,(k\,(\lambda x_1 : \alpha.\mathrm{in}_0\,(x_1))))$$

and not equal-up-to-parametricity to $1'$; the last fact follows from the remaining part of the proof.)

Now $f\left(d^{-1}\left(c^{-1}\,(1')\right)\right)$ is equal to $1'R$ up to parametricity; hence, $1'R$ is in the image of f. (Here "in image" is up to equations under parametricity.) On the other hand, in the types $((1 + R) \to R) \to R$ and $((R \to R + R) \to R) \to R$ there are only finite number (respectively 2^8 and $2^{\left(2^{16}\right)}$) of closed terms up to parametricity, so the extensional equality in the types are decidable, and we can compute—by programming—whether $1'R$ is in the image of f. Then the answer becomes false, hence we could show contradiction. $\square$

As above, the proof is given using computation by computer; a direct proof should be given.

Though the example in Section 3 comes from shortage of parametricity, the above example is a counterexample even under full-parametricity.

The above example denies the isomorphism in the term model with parametricity, but does not directly deny the isomorphism in other models of System F with parametricity. If we have some parametric models for which the isomorphism holds, it means that the logic of parametricity is not complete for such the models. Conversely thinking, if—as in the case of simply typed lambda calculus [2]—every "non-trivial" parametric model of System F forms a (singleton) class of models for which the logic of parametricity is complete, then the above counterexample works also for all such non-trivial models.

3 Tensor Products

Proposition 2 *The fibred endofunctor* $(-) \otimes (-)$ *on* $PFam(AP(D)_\perp)$ *does not preserve simple* Ω*-products.* $\square$

Proof. We here represent a per by the set of equivalent classes. Note that if the domains of considered pers are finite sets, then a relation, i.e., a regular subobject of the product of such two pers is just a relation between such two sets of equivalent classes such that the bottom class must be related to the bottom class.

Let us define an object $f = (f^p, f^r)$ over the generic object 1 in $\mathrm{PFam}\,(\mathrm{AP}\,(D)_\perp)$ as the following:

$$
\begin{aligned}
f^p(1) &:= \big\{\{\mathit{true}, \perp\}, \{\mathit{false}\}\big\} \\
f^p(R) &:= \big\{\{\mathit{true}\}, \{\perp, \mathit{false}\}\big\} \quad (\text{for } R \neq 1)
\end{aligned}
$$

$$
[d]\, f^r(A)\, [e] \overset{\text{def}}{\iff} \exists x \in \{\perp, \mathit{true}, \mathit{false}\}.\ x \in [d] \ \wedge\ x \in [e]
$$

$$
(\text{for } R, S \in |\mathrm{AP}\,(D)_\perp|,\ \text{and for regular subobject } A \rightarrowtail R \times S \text{ in } \mathrm{AP}\,(D)_\perp)\,.
$$

Now note that the fiber over 0—where $\prod f$ belongs—is nothing but $\mathrm{AP}\,(D)_\perp$ (up-to-iso), and also that the per part of a tensor product is determined by only the two per parts. Then

$$
\prod f = \{\{\perp\}, \{\mathit{true}\}, \{\mathit{false}\}\},
$$

and so

$$
\begin{aligned}
&\prod f \otimes \prod f \\
&= \big\{\{\langle \perp, \perp\rangle, \ldots\}, \{\langle \mathit{true}, \mathit{true}\rangle\}, \{\langle \mathit{true}, \mathit{false}\rangle\}, \{\langle \mathit{false}, \mathit{true}\rangle\}, \{\langle \mathit{false}, \mathit{false}\rangle\}\big\}.
\end{aligned}
$$

On the other hand, it can be easily checked by definition that

$$
\begin{aligned}
&\langle \mathit{true}, \mathit{false}\rangle \left(\prod (f \otimes f)\right) \langle \perp, \perp\rangle \ \text{ and} \\
&\langle \mathit{false}, \mathit{true}\rangle \left(\prod (f \otimes f)\right) \langle \perp, \perp\rangle\,,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\prod (f \otimes f) \\
&= \{\{\langle \perp, \perp\rangle, \langle \mathit{true}, \mathit{false}\rangle, \langle \mathit{false}, \mathit{true}\rangle, \ldots\}, \{\langle \mathit{true}, \mathit{true}\rangle\}, \{\langle \mathit{false}, \mathit{false}\rangle\}\}.
\end{aligned}
$$

Thus $\prod f \otimes \prod f$ and $\prod (f \otimes f)$ are not isomorphic. $\square$

Note that the above counterexample comes from (probably inevitable) shortage of parametricity of $\mathrm{PFam}\,(\mathrm{AP}\,(D)_\perp)$ in the sense that a term of a universal quantified type in $\mathrm{PFam}\,(\mathrm{AP}\,(D)_\perp)$ may be $\perp$ or else *dependently* on type instantiation.

Acknowledgements

I would like to thank Masahito Hasegawa and Shin-ya Katsumata for conjecturing the counterexample on continuation monads.

References

1. Kazuyuki Asada. Extensional universal types for call-by-value. In G. Ramalingam, editor, *Programming Languages and Systems*, volume 5356 of *Lecture Notes in Computer Science*, pages 122–137. Springer Berlin Heidelberg, 2008.
2. Alex K. Simpson. Categorical completeness results for the simply-typed lambda-calculus. In Mariangiola Dezani-Ciancaglini and Gordon Plotkin, editors, *Typed Lambda Calculi and Applications*, volume 902 of *Lecture Notes in Computer Science*, pages 414–427. Springer Berlin Heidelberg, 1995.