Automating Induction for Solving Horn Clauses

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Program Verification via Horn Constraint Solving

Verification Problems of Programs in **Various Paradigms** (e.g., functional [U.+ '08, '09, Rondon+ '08, ...], procedural [Grebenshchikov+ '12, Gurfinkel+ '15], object-oriented [Kahsai+ '16], multi-threaded [Gupta+ '11], constraint logic) with **Advanced Language Features** (e.g., algebraic data structures, linked data structures, exceptions, higher-order functions) with **Side-Effects** (e.g., non-termination, non-determinism, concurrency, assertions, destructive updates)

↓ Reduce

Horn Constraint Solving Problems

Overall Flow of Horn Constraint based Program Verification

$$P(x,0,0)$$

$$P(x,y,x+r) \Leftarrow P(x,y-1,r) \land y \neq 0$$

$$r \geq 0 \Leftarrow P(x,y,r) \land x \geq 0 \land y \geq 0$$

```
(* OCaml *)
let rec mult x y
  if y = 0 then 0
  else x + mult x (y - 1)
```

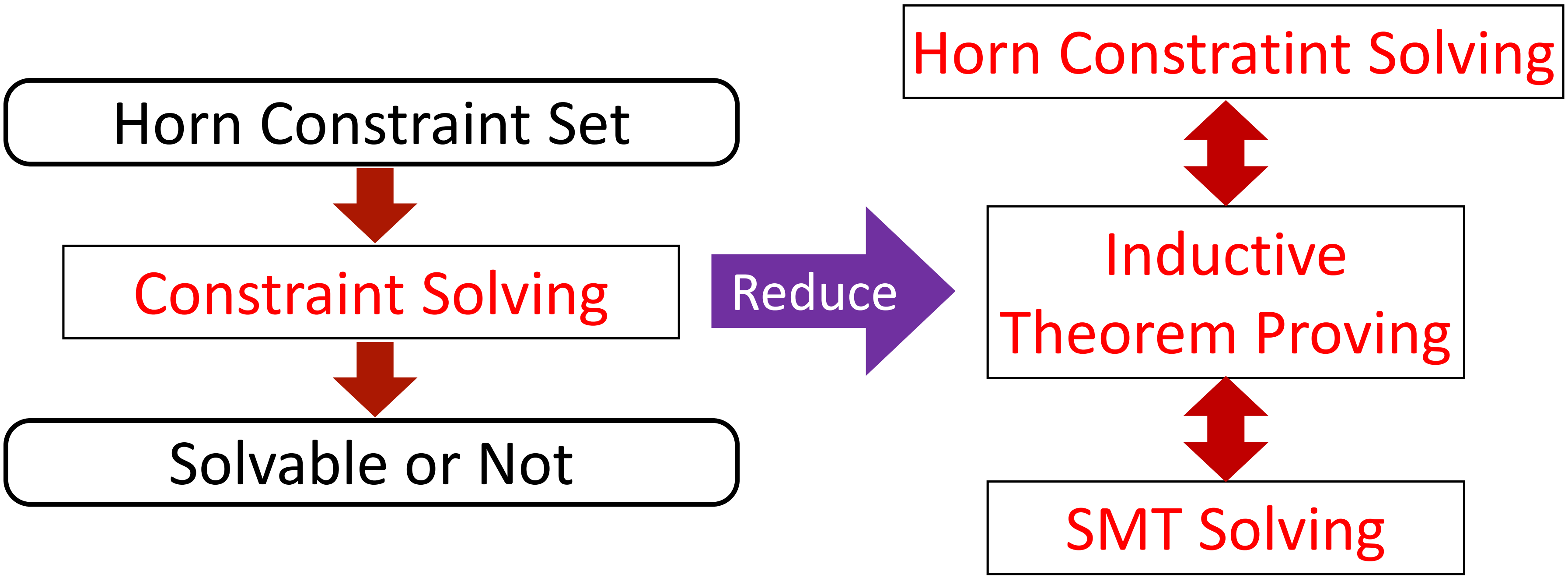
```
{- Haskell -}
mult :: Int -> Int -> Int
mult x 0 = 0
mult x y = x + mult x (y - 1)
```

```
  int s = 0;
  while(y != 0){

  return s;
}
```

$$P(x,y,r) \equiv r = x \times y$$

This Work

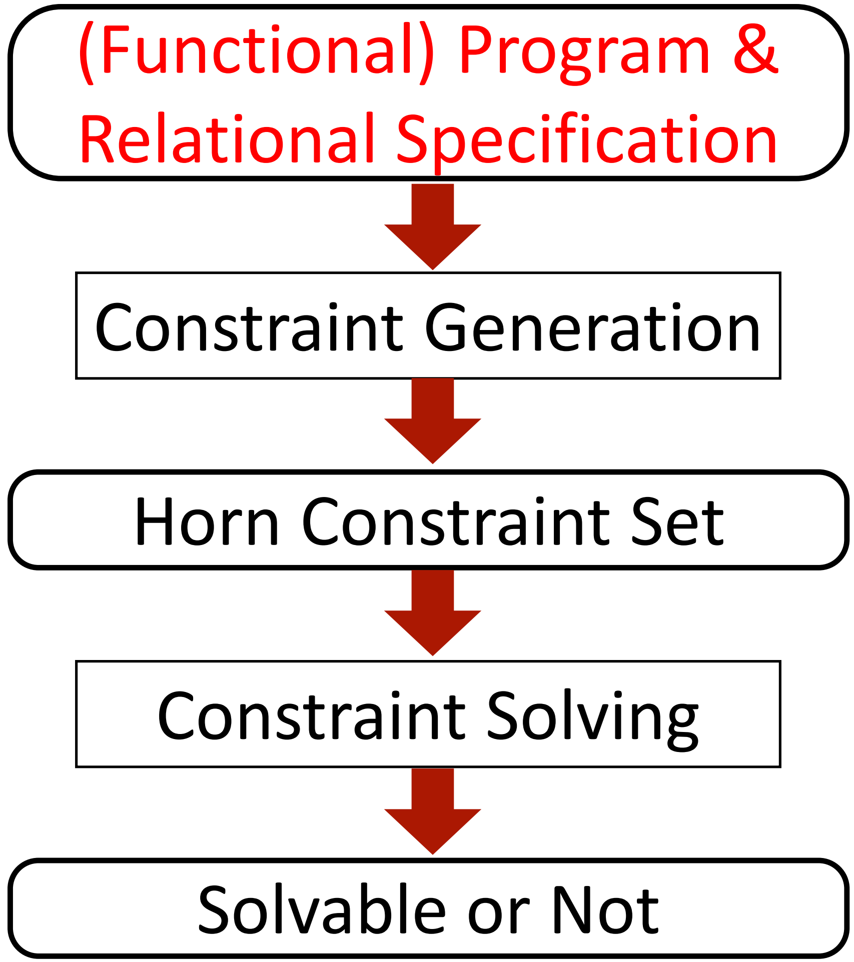


- Enable verification of ***relational specifications*** across programs in various paradigms
- Support constraints over any background theories (if the backend SMT solver does)

Relational Specifications

- Specifications that involve multiple function calls
  - Equivalence
  - Invertibility
  - Non-interference
  - Associativity
  - Commutativity
  - Distributivity
  - Monotonicity
  - Idempotency
  - …

Overall Flow of Horn Constraint based Program Verification



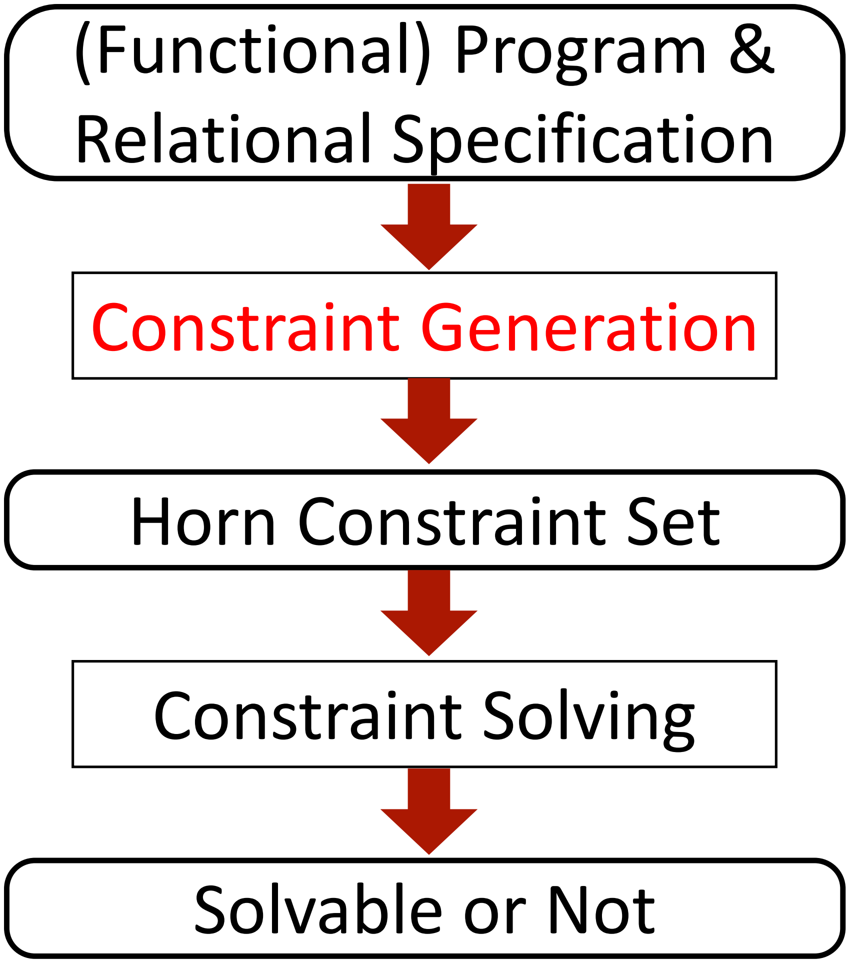
Example: (Functional) Program and Relational Specification

```
(* recursive function to compute "x × y" *)
let rec mult x y =
  if y = 0 then 0 else x + mult x (y - 1)

(* tail recursive function to compute "x × y + a" *)
let rec mult_acc x y a =
  if y = 0 then a else mult_acc x (y - 1) (a + x)

(* functional equivalence of mult and mult_acc *)
let main x y a = assert (mult x y + a = mult_acc x y a)
```

Overall Flow of Horn Constraint based Program Verification



Horn Constraint Generation [U.+ '09]

```
let rec mult x y =
  if y = 0 then 0
  else x + mult x (y - 1)
let rec mult_acc x y a =
  if y = 0 then a
  else mult_acc x (y - 1) (a + x)
```

```
let main x y a =
  assert (mult x y + a
              = mult_acc x y a)
```

$$P(x, 0, 0)$$

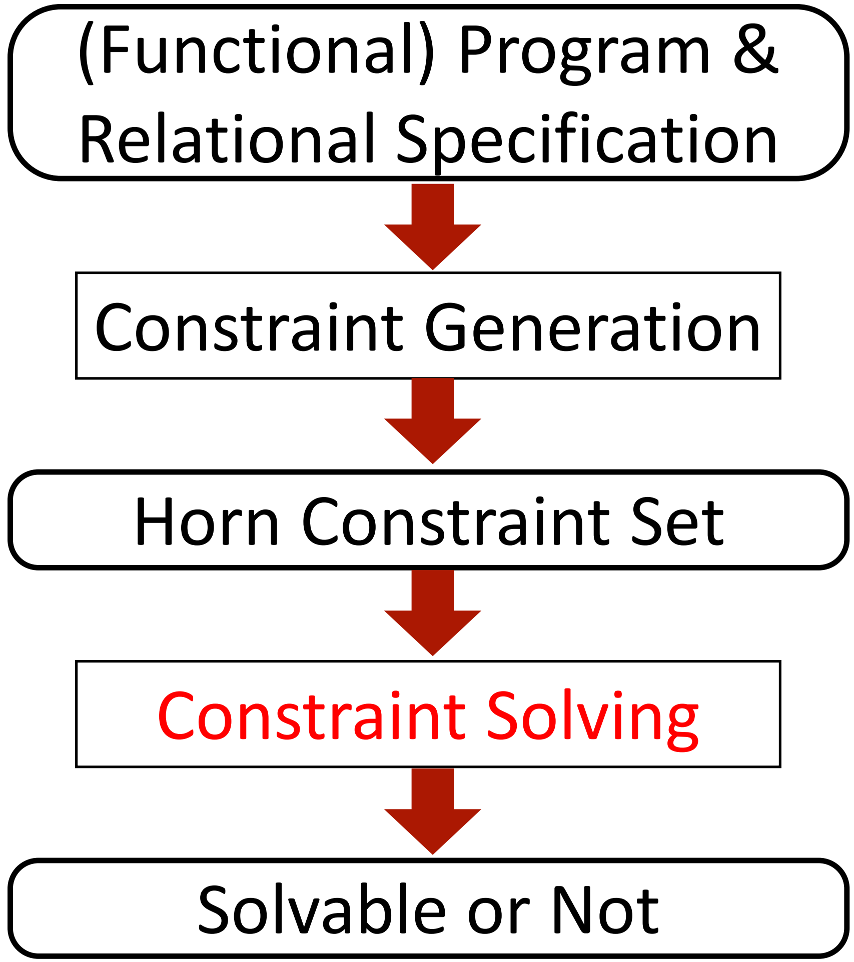
$$P(x, y, x + r) \Leftarrow P(x, y - 1, r) \wedge y \neq 0$$

$$Q(x, 0, a, a)$$

$$Q(x, y, a, r) \Leftarrow Q(x, y - 1, a + x, r) \wedge y \neq 0$$

$$s_1 + a = s_2 \Leftarrow P(x, y, s_1) \wedge Q(x, y, a, s_2)$$

Overall Flow of Horn Constraint based Program Verification



Horn Constraint Solving

- Check the existence of a solution for predicate variables satisfying all the Horn constraints
  - If a solution exists, the original program is guaranteed to satisfy the specification

Example (Non-relational) specification:

```
let main x y = if x >= 0 && y >= 0 then assert (mult x y >= 0)
```

$$P(x,0,0)$$
$$P(x,y,x+r) \Leftarrow P(x,y-1,r) \wedge y \neq 0$$
$$r \geq 0 \Leftarrow P(x,y,r) \wedge x \geq 0 \wedge y \geq 0$$

Solution 1: $P(x,y,r) \equiv x \geq 0 \wedge y \geq 0 \Rightarrow r \geq 0$ — QF-LIA

Solution 2: $P(x,y,r) \equiv r = x \times y$ — QF-NIA

Nonlinear

Previous Methods for Solving Horn Clause Constraints [U.+ ’08,’09, Rondon+ ’08, Gupta+ ’11, Hoder+ ’11,’12, McMillan+ ’13, Rümmer+ ’13, ...]

Find a solution expressible in QF-LIA (or QF-LRA)

$$P(x,0,0)$$
$$P(x,y,x+r) \Leftarrow P(x,y-1,r) \wedge y \neq 0$$
$$r \geq 0 \Leftarrow P(x,y,r) \wedge x \geq 0 \wedge y \geq 0$$

Solution 1: $P(x,y,r) \equiv x \geq 0 \wedge y \geq 0 \Rightarrow r \geq 0$

QF-LIA

~~Solution 2: $P(x,y,r) \equiv r = x \times y$~~

QF-NIA

Example Constraints that Can Not be Solved by Previous Methods

Constraint Solving Fails!

$$P(x, 0, 0)$$

$$P(x, y, x + r) \Leftarrow P(x, y \ldots$$

$$Q(x, 0, a, a)$$

$$Q(x, y, a, r) \Leftarrow Q(x, y - 1, a + x, r) \wedge y \neq 0$$

$$s_1 + a = s_2 \Leftarrow P(x, y, s_1) \wedge Q(x, y, a, s_2)$$

QF-NIA

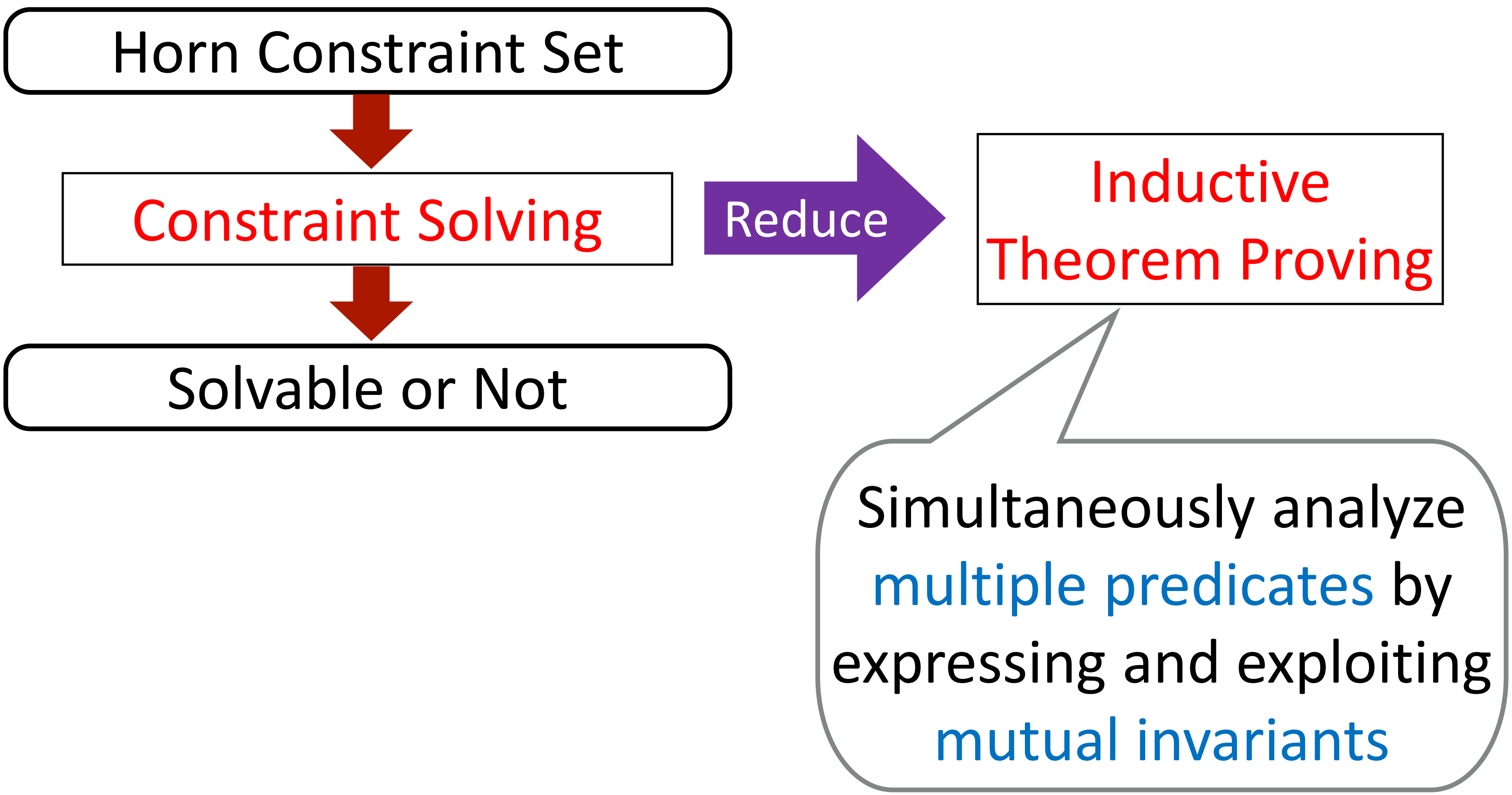
Analyzed separately from Q

Analyzed separately from P

$P(x, y, s_1) \equiv s_1 = x \times y$

$Q(x, y, a, s_2) \equiv s_2 = x \times y + a$

Our Constraint Solving Method



Reduction from Constraint Solving to Inductive Theorem Proving

$P(x,0,0) \quad P(x,y,x+r) \Leftarrow P(x,y-1,r) \land y \neq 0$
$Q(x,0,a,a) \quad Q(x,y,a,r) \Leftarrow Q(x,y-1,a+x,r) \land y \neq 0$
$s_1 + a = s_2 \Leftarrow P(x,y,s_1) \land Q(x,y,a,s_2)$

$$\frac{\models y = 0 \land r = 0}{P(x,y,r)} \quad \frac{P(x,y-1,r-x) \quad \models y \neq 0}{P(x,y,r)}$$

$$\frac{\models y = 0 \land a = r}{Q(x,y,a,r)} \quad \frac{Q(x,y-1,a+x,r) \quad \models y \neq 0}{Q(x,y,a,r)}$$

Prove this by induction on derivation of $P(x,y,s_1)$, $\mathrm{Q}(\mathrm{x},\mathrm{y},\mathrm{s}_2)$

$\forall x,y,s_1,a,s_2.\, P(x,y,s_1) \land Q(x,y,a,s_2) \Rightarrow s_1 + a = s_2$

Principle of Induction on Derivation

$$\forall D.\ \psi(D) \quad \text{if and only if}$$
$$\forall D.\ \big(\forall D'.\, D' \prec D \Rightarrow \psi(D')\big) \Rightarrow \psi(D)$$

where $D' \prec D$ represents that
D' is a strict sub-derivation of D

$$D = \cfrac{\cfrac{\cfrac{D_1}{J_3} \quad D_2}{J_2} \quad D_3 \quad \cfrac{D_4}{J_4}}{J_1}$$

Assume
$\psi(D_1), \psi(D_2)$,
$\psi(D_3), \psi(D_4)$
and prove $\psi(D)$

Horn Constraint Solving:

$$P(x, 0, 0)$$

$$P(x, y, x + r) \Leftarrow P(x, y - 1, r) \wedge y \neq 0$$

$$Q(x, 0, a, a)$$

$$Q(x, y, a, r) \Leftarrow Q(x, y - 1, a + x, r) \wedge y \neq 0$$

$$s_1 + a = s_2 \Leftarrow P(x, y, s_1) \wedge Q(x, y, a, s_2)$$

Induction hypotheses and lemmas

Judgment

$$\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2$$

Premises

$$\frac{\models y = 0 \wedge \quad}{P(x, y, r)} \qquad \frac{\quad, y - 1, r - x) \qquad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \wedge a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y - 1, a + x, r) \qquad \models y \neq 0}{Q(x, y, a, r)}$$

$$\frac{\models y = 0 \land r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \qquad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \land a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \qquad \models y \neq 0}{Q(x, y, a, r)}$$

Add an induction hypothesis

Guard to avoid unsound application

$$\gamma = \begin{array}{l} \forall x', y', s_1', a', s_2'. D\left(P(x', y', s_1')\right) \prec D\left(P(x, y, s_1)\right) \land \\ \quad P(x', y', s_1') \land Q(x', y', a', s_2') \Rightarrow s_1' + a' = s_2' \end{array}$$

Induct

Unfold

Case analysis on the last rule used

$$\frac{\gamma; \cdots, y = 0 \land s_1 = 0 \vdash \cdots \qquad \gamma; \cdots, P(x, y-1, s_1 - x), y \neq 0 \vdash \cdots}{\emptyset; \underline{P(x, y, s_1)}, Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \land r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \land a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x, y, a, r)}$$

$$\frac{\gamma; \cdots, y = 0 \land s_1 = 0 \vdash \cdots}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \land r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x, y, r)}$$

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Case analysis on the last rule used

Unfold

$$\frac{\dfrac{\gamma; \cdots, \cdots \land y = 0 \land a = s_2 \vdash \cdots \qquad \gamma; \cdots, Q(x, y-1, a+x, s_2), \cdots \land y \neq 0 \vdash \cdots}{\gamma; P(x, y, s_1), \underline{Q(x, y, a, s_2)}, y = 0 \land s_1 = 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \land r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \land a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x, y, a, r)}$$

$$\cfrac{\cfrac{\gamma; \cdots, \cdots \land y = 0 \land a = s_2 \vdash \cdots}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), y = 0 \land s_1 = 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

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$$\frac{\models y = 0 \land a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x, y, a, r)}$$

Validity checking

Valid

$$\cfrac{\cfrac{\cfrac{\models y = 0 \land s_1 = 0 \land a = s_2 \Rightarrow s_1 + a = s_2}{\gamma; \cdots, y = 0 \land s_1 = 0 \land a = s_2 \vdash s_1 + a = s_2}}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), y = 0 \land s_1 = 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \wedge r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x, y, r)}$$

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$$\frac{\dfrac{\gamma; \cdots, Q(x, y-1, a+x, s_2), \cdots \wedge y \neq 0 \vdash \cdots}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), y = 0 \wedge s_1 = 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \land r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \qquad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \land a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \qquad \models y \neq 0}{Q(x, y, a, r)}$$

Valid

$$\cfrac{\cfrac{\cfrac{\models y = 0 \land s_1 = 0 \land y \neq 0 \Rightarrow s_1 + a = s_2}{\gamma; \cdots, Q(x, y-1, a+x, s_2), y = 0 \land s_1 = 0 \land y \neq 0 \vdash s_1 + a = s_2}}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), y = 0 \land s_1 = 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

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$$\frac{\models y = 0 \wedge a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x, y, a, r)}$$

$$\frac{\gamma; \cdots, P(x, y-1, s_1 - x), y \neq 0 \vdash \cdots}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \land r = 0}{P(x,y,r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x,y,r)}$$

$$\frac{\models y = 0 \land a = r}{Q(x,y,a,r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x,y,a,r)}$$

Unfold

Case analysis on the last rule used

$$\frac{\dfrac{\gamma; \cdots, \cdots \land y = 0 \land a = s_2 \vdash \cdots \qquad \gamma; \cdots, Q(x, y-1, a+x, s_2), \cdots \land y \neq 0 \vdash \cdots}{\gamma; P(x,y,s_1), \underline{Q(x,y,a,s_2)}, {\color{red}P(x, y-1, s_1 - x)}, y \neq 0 \vdash s_1 + a = s_2}}{\emptyset; P(x,y,s_1), Q(x,y,a,s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \wedge r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \wedge a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x, y, a, r)}$$

$$\frac{\dfrac{\gamma; \cdots, \cdots \wedge y = 0 \wedge a = s_2 \vdash \cdots}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), P(x, y-1, s_1 - x), y \neq 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

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Valid

$$\cfrac{\cfrac{\cfrac{\models y \neq 0 \wedge y = 0 \wedge a = s_2 \Rightarrow s_1 + a = s_2}{\gamma; \cdots, y \neq 0 \wedge y = 0 \wedge a = s_2 \vdash s_1 + a = s_2}}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), P(x, y-1, s_1 - x), y \neq 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \wedge r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \wedge a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x, y, a, r)}$$

$$\frac{\dfrac{\gamma; \cdots, Q(x, y-1, a+x, s_2), \cdots \wedge y \neq 0 \vdash \cdots}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), P(x, y-1, s_1 - x), y \neq 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \land r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \quad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \land a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \quad \models y \neq 0}{Q(x, y, a, r)}$$

$$\sigma(\gamma) = \begin{array}{c} D(P(x, y-1, s_1 - x)) \prec D(P(x, y, s_1)) \land P(x, y-1, s_1 - x) \land \\ Q(x, y-1, a+x, s_2) \Rightarrow (s_1 - x) + (a + x) = s_2 \end{array}$$

IndHyp Apply induction hypothesis

$$\frac{\frac{\frac{\gamma; \cdots, y \neq 0 \land (s_1 - x) + (a + x) = s_2 \vdash s_1 + a = s_2}{\gamma; \cdots, P(x, y-1, s_1 - x), Q(x, y-1, a+x, s_2), y \neq 0 \vdash s_1 + a = s_2}}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), P(x, y-1, s_1 - x), y \neq 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

$$\frac{\models y = 0 \wedge r = 0}{P(x, y, r)} \qquad \frac{P(x, y-1, r-x) \qquad \models y \neq 0}{P(x, y, r)}$$

$$\frac{\models y = 0 \wedge a = r}{Q(x, y, a, r)} \qquad \frac{Q(x, y-1, a+x, r) \qquad \models y \neq 0}{Q(x, y, a, r)}$$

Valid

$$\cfrac{\cfrac{\cfrac{\cfrac{\models y \neq 0 \wedge (s_1 - x) + (a + x) = s_2 \Rightarrow s_1 + a = s_2}{\gamma; \cdots, y \neq 0 \wedge (s_1 - x) + (a + x) = s_2 \vdash s_1 + a = s_2}}{\gamma; \cdots, P(x, y-1, s_1 - x), Q(x, y-1, a+x, s_2), y \neq 0 \vdash s_1 + a = s_2}}{\gamma; P(x, y, s_1), Q(x, y, a, s_2), P(x, y-1, s_1 - x), y \neq 0 \vdash s_1 + a = s_2}}{\emptyset; P(x, y, s_1), Q(x, y, a, s_2) \vdash s_1 + a = s_2}$$

QED

Properties of Inductive Proof System for Horn Constraint Solving

- Soundness: If the goal is proved, the original Horn constraints have a solution (which may not be expressible in the underlying logic)
- *Relative* Completeness: If the original constraints have a solution *expressible in the underlying logic*, the goal is provable

Automating Induction

- Use the following rule application strategy:
  - Repeatedly apply INDHYP until no new premises are added
  - Apply VALID whenever a new premise is added
  - Select some $P(\tilde{t})$ and apply INDUCT and UNFOLD
- Close a proof branch by using:
  - SMT solvers: provide efficient and powerful reasoning about **data structures** (e.g., integers, reals, algebraic data structures) but predicates are abstracted as uninterpreted functions
  - Horn constraint solvers: provide bit costly but powerful reasoning about **inductive predicates**

Prototype Constraint Solver

- Use **Z3** and ***μ*Z PDR** engine respectively as the backend SMT and Horn constraint solvers
- Integrated with a refinement type based verification tool **RCaml** for the OCaml functional language
- Can exploit lemmas which are:
  - User-supplied,
  - Heuristically obtained from the given constraints, or
  - Automatically generated by an abstract interpreter
- Can generate a counterexample (if any)

Experiments on IsaPlanner Benchmark Set

- 85 (mostly) relational verification problems of total functions on inductively defined data structures

Inductive Theorem Prover	#Successfully Proved
RCaml	68
Zeno	82 [Sonnex+ '12]
HipSpec	80 [Claessen+ '13]
CVC4	80 [Reynolds+ '15]
ACL2s	74 (according to [Sonnex+ '12])
IsaPlanner	47 (according to [Sonnex+ '12])
Dafny	45 (according to [Sonnex+ '12])

Support automatic lemma discovery & goal generalization

Experiments on Benchmark Programs with Advanced Language Features & Side-Effects

- 30 (mostly) relational verification problems for:
  - Complex integer functions: Ackermann, McCarthy91
  - Nonlinear real functions: dyn_sys
  - Higher-order functions: fold_left, fold_right, repeat, find, ...
  - Exceptions: find
  - Non-terminating functions: mult, sum, …
  - Non-deterministic functions: randpos
  - Imperative procedures: mult_Ccode

ID	specification	kind	features	result	time (sec.)
1	$\texttt{mult}\ x\ y + a = \texttt{mult\_acc}\ x\ y\ a$	equiv	P	✓	0.378
2	$\texttt{mult}\ x\ y = \texttt{mult\_acc}\ x\ y\ 0$	equiv	P	✓†	0.803
3	$\texttt{mult}\ (1+x)\ y = y + \texttt{mult}\ x\ y$	equiv	P	✓	0.403
4	$y \geq 0 \Rightarrow \texttt{mult}\ x\ (1+y) = x + \texttt{mult}\ x\ y$	equiv	P	✓	0.426
5	$\texttt{mult}\ x\ y = \texttt{mult}\ y\ x$	comm	P	✓‡	0.389
6	$\texttt{mult}\ (x+y)\ z = \texttt{mult}\ x\ z + \texttt{mult}\ y\ z$	dist	P	✓	1.964
7	$\texttt{mult}\ x\ (y+z) = \texttt{mult}\ x\ y + \texttt{mult}\ x\ z$	dist	P	✓	4.360
8	$\texttt{mult}\ (\texttt{mult}\ x\ y)\ z = \texttt{mult}\ x\ (\texttt{mult}\ y\ z)$	assoc	P	✗	n/a
9	$0 \leq x_1 \leq x_2 \land 0 \leq y_1 \leq y_2 \Rightarrow \texttt{mult}\ x_1\ y_1 \leq \texttt{mult}\ x_2\ y_2$	mono	P	✓	0.416
10	$\texttt{sum}\ x + a = \texttt{sum\_acc}\ x\ a$	equiv		✓	0.576
11	$\texttt{sum}\ x = x + \texttt{sum}\ (x-1)$	equiv		✓	0.452
12	$x \leq y \Rightarrow \texttt{sum}\ x \leq \texttt{sum}\ y$	mono		✓	0.593

- 28 (2 required lemmas) successfully proved by **RCaml**

- 3 proved by Horn constraint solver ***μ*Z PDR**
- 2 proved by inductive theorem prover **CVC4** (if inductive predicates are encoded using uninterpreted functions)

ID	specification	kind	features	result	time (sec.)
24	$\texttt{noninter}\ h_1\ l_1\ l_2\ l_3 = \texttt{noninter}\ h_2\ l_1\ l_2\ l_3$	nonint	P	✓	1.203
25	$\texttt{try find\_opt}\ p\ l = \texttt{Some (find}\ p\ l\texttt{) with}$ $\texttt{Not\_Found} \rightarrow \texttt{find\_opt}\ p\ l = \texttt{None}$	equiv	H, E	✓	1.065
26	$\texttt{try mem (find ((=)}\ x\texttt{)}\ l\texttt{)}\ l\ \texttt{with Not\_Found} \rightarrow \neg(\texttt{mem}\ x\ l)$	equiv	H, E	✓	1.056
27	$\texttt{sum\_list}\ l = \texttt{fold\_left}\ (+)\ 0\ l$	equiv	H	✓	6.148
28	$\texttt{sum\_list}\ l = \texttt{fold\_right}\ (+)\ l\ 0$	equiv	H	✓	0.508
29	$\texttt{sum\_fun randpos}\ n > 0$	equiv	H,D	✓	0.319
30	$\texttt{mult}\ x\ y = \texttt{mult\_Ccode}(x, y)$	equiv	P, C	✓	0.303

† A lemma $P_{\texttt{mult\_acc}}(x,y,a,r) \Rightarrow P_{\texttt{mult\_acc}}(x,y,a-x,r-x)$ is used

‡ A lemma $P_{\texttt{mult}}(x,y,r) \Rightarrow P_{\texttt{mult}}(x-1,y,r-y)$ is used

Used a machine with Intel(R) Xeon(R) CPU (2.50 GHz, 16 GB of memory).

Conclusion

- Proposed an automated verification method combining Horn constraint solving and inductive theorem proving
  - Enable relational verification across programs in various paradigms with advanced language features and side-effects
  - Support constraints over any background theories (if the backend SMT solver does)
- Future and ongoing work:
  - Automatic lemma discovery and goal generalization
  - Relational program synthesis
  - Coinduction